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Techniques of Integration

Definite integral $\int_a^b f(x)\,dx$

Indefinite integral $\int f(x)\,dx$ (answer is the general antiderivative)

Example: $\int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{(1)^3}{3} - \frac{(0)^3}{3} = \frac{1}{3}$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

Recall some rules we figured out:

$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$ $\qquad$ $\int \sec x \tan x\,dx = \sec x + C$

$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln x + C$ $\qquad$ $\int e^x\,dx = e^x + C$

$\int \sin x\,dx = -\cos x + C$

$\int \cos x\,dx = \sin x + C$

$\int \sec^2 x\,dx = \tan x + C$

Consider $\int 2x\sqrt{1+x^2}\,dx$

this looks hard - we need to simplify somehow

Introduce a new variable to simplify the expression

Let $u = 1 + x^2$

Notice $\frac{du}{dx} = 2x$ move around dx to get $du = 2x\,dx$

So,

$$\int 2x\sqrt{1+x^2}\,dx = \int \underbrace{\sqrt{1+x^2}}_{u}\,\underbrace{2x\,dx}_{du} = \int \sqrt{u}\,du = \int u^{1/2}\,du$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \boxed{\frac{2}{3}(1+x^2)^{3/2} + C}$$

What just happened??

This process is the opposite of the Chain Rule

Check $\frac{2}{3}(1+x^2)^{3/2} + C$ is correct

$$\frac{d}{dx}\left(\frac{2}{3}(1+x^2)^{3/2} + C\right) = \frac{3}{2}\cdot\frac{2}{3}(1+x^2)^{1/2}\cdot 2x$$

$$= 2x\sqrt{1+x^2}$$

When we use chain rule:

$$\int f'(\underbrace{g(x)}_{\text{you have a funcon inside}})\underbrace{g'(x)}_{\text{the derivative of the inside is multiplied.}}\,dx$$

Ex: (1) $\int x^3 \cos(x^4+2)\,dx$

$u = x^4+2 \qquad \frac{du}{dx} = 4x^3$ so $du = 4x^3\,dx$

$$\int \cos(\underbrace{x^4+2}_{u}) \cdot \underbrace{x^3\,dx}_{du/4} = \int \cos u \cdot \frac{du}{4}$$

$$= \frac{1}{4}\int \cos u\,du$$

$$= \frac{1}{4}(\sin u + C)$$

$$= \frac{1}{4}(\sin(x^4+2) + C)$$

This is called u-substitution.

The Process:

(1) decide possibilities for what u equals

(2) calculate du

(3) sub du and u into integral. There should be no x's left!

(4) take resulting integral

(5) unsubstitute (put x's back in)

Ex: $\int \frac{1}{4x+2}\,dx$ $u = 4x+2$

$\frac{du}{dx} = 4 \Leftrightarrow du = 4\,dx$

$$\int \frac{1}{u}\frac{du}{4} = \frac{1}{4}\ln u + C$$
$$= \frac{1}{4}\ln(4x+2) + C$$

Definite Integrals & Substitution

One Method: take integral as above then plug in the bounds.

Another... sort of shorter method.

$$\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f'(u)\,du$$

let $u = g(x)$

$du = g'(x)\,dx$

Example: $\int_0^4 \sqrt{2x+1}\,dx$ $u = 2x+1$

$du = 2\,dx$

$$\int_{2(0)+1}^{2(4)+1} \sqrt{u}\,\frac{du}{2}$$

$$= \int_1^9 \frac{1}{2}\sqrt{u}\,du = \frac{2}{3}\cdot\frac{1}{2}u^{3/2}\Big|_1^9 = \frac{1}{3}9^{3/2} - \frac{1}{3}1^{3/2}$$
$$= 9 - \frac{1}{3} = \frac{26}{3}$$

Why does that work?

$$\int_a^b \underbrace{f'(g(x))g'(x)}\,dx$$

Know antider is $f(g(x))$

$$\int_{g(a)}^{g(b)} f'(u)\,du = f(u)\Big|_{g(a)}^{g(b)}$$

$$= f(g(b)) - f(g(a))$$

Ex: $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$ $\qquad u = \ln x$, $du = \frac{1}{x}dx$

$$= \int_0^1 u\,du$$

$$= \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$